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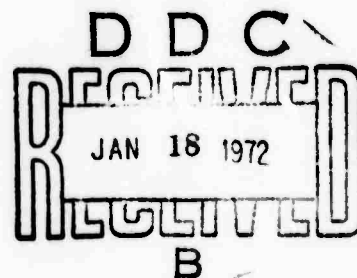
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## 13. ABSTRACT

The practical consequences of the commonly occurring thermospheric condition of turning points for propagating acoustic and internal gravity wave modes have been investigated. This has required some rather novel theoretical developments on the W.K.B. approximation, turning points, and the measurement of phase velocities. In the process of carrying out this investigation some sample calculations were made illustrating the relationships between phase velocity measurements and the existing values of the Vaisala frequency and acoustic cut-offs. A study was also made of the reflection of internal gravity waves of long period at height heights for which the wavelength becomes of the order of the mean free path of the molecules. The net effect is one of damping on the long-range propagation of trapped internal waves.

The program of research offered in the work proposal for F.Y. 1971 (December 1, 1970 - November 30, 1971) has been fulfilled as follows:

- 1) We have investigated the practical consequences of the commonly occurring thermospheric condition of turning points for propagating acoustic and internal gravity wave modes. This has required some rather novel theoretical developments, which we have entitled : "The W.K.B. Approximation, Turning Points and the Measurement of Phase Velocities" (see below).
- 2) In the process of carrying out 1), we have made some sample calculations illustrating the relationships between phase velocity measurements and the existing values of the Vaisala frequency and acoustic cut-offs (figures 1, 2, 3).
- 3) We have also included a very brief and probably oversimplified study of the reflection of internal gravity waves of long period at great heights for which the wavelength becomes of the order of the mean free path of the molecules. Since the net effect is one of damping on the long-range propagation of trapped internal waves, we have entitled this report "Damping of Very Long Internal Gravity Waves in the Atmosphere" (partly carried over from F.Y. 1970).

1

THE W.K.B. APPROXIMATION, TURNING POINTS AND  
THE MEASUREMENT OF PHASE VELOCITIES

1. Introduction

In the fields of acoustic<sup>(1,2)</sup> and electromagnetic<sup>(3,4)</sup> wave theory, as well as in some aspects of elastic wave theory<sup>(5)</sup>, W.K.B. type methods often offer the only sensible means for obtaining useful, closed-form approximations for the description of wavefields in stratified media. The method is, in many respects, ideally suited for the calculation of the far field in wave ducts<sup>(1)</sup>, of diffracted fields under conditions of caustic formation<sup>(2,6)</sup>, of focalization intensities, etc. Errors incurred by the use of this method are often no greater than experimental uncertainties, such as those due to fluctuations of the medium<sup>(1)</sup>; indeed, were it not for the fact that the approximation diverges near turning points, the method would be quite sufficient to deal with most practical problems and the calculation of wavefields by means of exact solutions of the wave equation would acquire a largely eclectic interest. Thus, in long range transmission calculations in geoacoustics and in e.m. theory, the cumulative experimental uncertainties of phase are often comparable to or greater than the errors of the W.K.B. procedure: as long as one can stay away from turning point effects, there is then no reason to try to improve upon this approximation.

There exists, however, another class of problems in which one tries to make accurate phase comparisons over short ranges. This occurs, for example, in the design or interpretation of array measurements (i.e., of any system consisting of two or more sensors comparing phase). The usual methods of relating phases for neighboring measurements are equivalent to the tacit assumption that the medium is locally homogeneous or that ray optics and the W.K.B. picture are valid; while this assumption is fre-

quently justified, it can and will break down in stratified geophysical media in the neighborhood of turning points. Thus, in a vertically stratified medium (in which the physical parameters are functions of the vertical  $z$  coordinate), this breakdown may lead to false conclusions concerning vertical phase velocity and directionality. These observations, which we shall clarify in sections 3, 4 and 5, are germane, in principle, to geoacoustic fields like underwater sound, atmospheric infrasonics and e.m. waves. As we shall see, they are probably most pronounced for long period acoustic-gravity waves in the atmosphere.

## 2. The W.K.B. method and the turning point phenomenon: review

Consider a stratified medium in which the physical parameters are functions of  $z$  only. Whether we are dealing with acoustic, gravity or e.m. waves, the assumption of simple harmonic wavetrains, i.e., of solutions proportional to  $e^{i(\alpha x - \omega t)}$  reduces the problem to a Helmholtz equation of the type:

$$\frac{d^2 h}{dz^2} + \gamma^2 h = 0 \quad (1)$$

where  $\gamma$  is some function of  $z$ .

The classic W.K.B. approximation has the form:

$$h = \gamma^{-1/2} e^{i(s-s_0)} \quad (2)$$

with

$$s = \pm \int^z \gamma dz \quad (3)$$

and  $s_0$  is an integration constant.

As is well known<sup>(1,2)</sup>, equation 2 is a good approximation as long as:

$$\frac{1}{4} \left| \frac{1}{\gamma^2} \frac{d^2}{dz^2} \ln \gamma^2 - \frac{1}{\gamma^2} \left( \frac{d}{dz} \ln \gamma \right)^2 \right| \ll 1 \quad (4)$$

a condition that is seen to break down near

$$\gamma = 0 \quad (5)$$

i.e., at a turning point of the Helmholtz equation (1). With the exception of the neighborhood of multiple turning points (for which one or more derivatives of  $\gamma$  also vanish), (4) is equivalent to:

$$\frac{1}{\gamma} \left| \frac{d}{dz} \ln \gamma \right| \ll 1 \quad (6)$$

This somewhat more succinct statement says that the W.K.B. approximation applies as long as the physical properties of the medium do not change appreciably over a vertical wavelength; it may also be shown<sup>(7)</sup> that this is equivalent to assuming that, as the wave energy propagates through the medium, one may neglect the process of continuous backscatter (reflection) from the stratification. Near the turning point itself, the  $z$  wavelength becomes very large, conditions (4) and (6) break down and the approximation (2) becomes divergent. Exact solutions near the turning point of course show no unusual behavior at  $\gamma = 0$ : this point simply marks the transition from a region of oscillating solutions ( $\gamma^2 > 0$ ) to exponential ones ( $\gamma^2 < 0$ ): the wave function and all its derivatives are continuous through the turning point.

It is useful and instructive to examine the case of a linear  $\gamma^2$ :

$$\gamma^2 = Pz + \gamma_0^2 \quad (7)$$

in which case,

$$s = \pm \frac{2}{3P} \gamma^3, \quad \text{for } \gamma^2 > 0 \quad (8)$$

Standing wave or modal solutions may then be expressed in terms of Bessel functions:

$$h = \gamma [A J_{1/3}(s) + B J_{-1/3}(s)] \quad (9)$$

$$\frac{dh}{dz} = \gamma^2 [A J_{2/3}(s) - B J_{2/3}(s)] \quad (10)$$

and, in the  $\gamma^2 < 0$  region:

$$\gamma'^2 = -\gamma^2 \quad (11)$$

$$\sigma = \frac{2}{3\gamma} \gamma'^3$$

$$h = \gamma' [-AI_{1/3}(\sigma) + BI_{-1/3}(\sigma)] \quad (12)$$

$$\frac{dh}{dz} = \gamma'^2 [AI_{-2/3}(\sigma) - BI_{2/3}(\sigma)] \quad (13)$$

where  $I_{\pm 1/3}$ ,  $I_{\pm 2/3}$  are the modified Bessel functions.

It is a simple matter to verify that the W.K.B. approximation corresponds to the asymptotic forms of (9), (12) for large  $s$  or  $\sigma$ , an observation that provides a direct and classic method for determining the connection formulas between W.K.B. solutions to both sides of the turning point<sup>(8)</sup>.

The progressive wave solutions corresponding to (9), (10) have the form:

$$h = \gamma H_{1/3}^{(1,2)}(s) \quad (14)$$

$$\frac{dh}{dz} = \gamma^2 H_{-2/3}^{(1,2)}(s) \quad (15)$$

where Hankel functions of the first or second kind are used, depending upon whether one wishes to represent down- or up-traveling disturbances (depending upon the sign of the exponent in  $e^{*i\omega t}$ ). This is conveniently seen by going to the asymptotic forms:

$$H_{1/3}^{(1)} \approx 2^{1/2} (\pi s)^{-1/2} e^{i(s - 5\pi/12)} \quad (16)$$

$$H_{1/3}^{(2)} \approx 2^{1/2} (\pi s)^{-1/2} e^{-i(s - 5\pi/12)} \quad (17)$$

We see that the W.K.B. approximation (2) corresponds to the use of



these asymptotic approximations. Equations (16), (17) are in fact quite accurate for  $s \gg 1$  - a condition which, by virtue of (8) and (7) is equivalent to the statement  $\frac{1}{\gamma} \left| \frac{d}{dz} \ln \gamma \right| \ll \frac{1}{3}$ , i.e., is consistent with (6).

In dealing with a medium of indefinite extent and linear  $\gamma^2$ , we require that in the region  $\gamma^2 < 0$  solutions (12), (13) give a convergent energy density for  $\sigma \rightarrow \infty$ . This is achieved by taking  $A = B$ , giving for  $h$ :

$$h = A 3^{\frac{1}{2}} \pi^{-1} \gamma^{-1} K_{1/3}(\sigma) \quad (18)$$

where  $K_{1/3}(\sigma)$  vanishes like  $\sigma^{-\frac{1}{2}} e^{-\sigma}$  for large  $\sigma$ .

Note that the standing wave solutions (9) with  $A = B$ , together with (18), are the result of superposing an incident and a totally reflected field. Thus, taking the turning point  $\gamma = 0$  at the origin  $z = 0$ , with  $\gamma^2 > 0$  for  $z < 0$ , we have  $P < 0$ , and taking  $s = -\int^z \gamma dz = -\frac{2}{3P} \gamma^3$  gives  $s > 0$  for  $\gamma^2 > 0$ ; with these conventions, the incident wavetrain traveling towards the turning point would be

$$\phi_i = A \gamma^{\frac{1}{2}} \mathcal{H}_{1/3}^{(2)}(s) e^{-i\omega t} \quad (19)$$

and the reflected wave train, traveling away from the turning point:

$$\phi_R = A e^{i\frac{\pi}{3}} \gamma^{\frac{1}{2}} \mathcal{H}_{1/3}^{(1)}(s) e^{-i\omega t} \quad (20)$$

It is easily verified that the superposition of (19) and (20) gives the standing wavefield (9) with  $A = B$ , and gives (18) in the  $\gamma^2 < 0$  region. It is important to understand here that this total reflection process is spread over a zone near  $\gamma = 0$ . Thus, as  $s \rightarrow 0$  we approach the turning point and  $\phi_R \rightarrow -e^{i\frac{\pi}{3}} \phi_i$ , so that the actual reflection coefficient at  $z = 0$  is equal to  $-e^{i\frac{\pi}{3}} = e^{-2i\frac{\pi}{3}}$ ; on the other hand, as  $s \rightarrow \infty$ , we see from (16), (17) that  $\phi_R \rightarrow e^{2i(s - \pi/4)} \phi_i$ , i.e., the effective reflection coefficient is asymptotically equal to  $e^{-i\frac{\pi}{2}}$ . There is then a region, in the

vicinity of the turning point, that contributes an extra phase change of  $\pi/6$ , above and beyond the integrated phase path contribution: in other words, the reflection process is not concentrated at the turning point, but is smeared out over a layer of thickness given by  $s \approx 1$ .

It is also worth noting that while  $h$  and  $dh/dz$  both change from oscillating to exponential type solutions, they do not obey the same Helmholtz equation<sup>(9)</sup>. Thus, differentiating (1) gives:

$$\frac{d^3 h}{dz^3} + \gamma^2 \frac{dh}{dz} + h \frac{d\gamma^2}{dz} = 0 \quad (21)$$

Taking

$$y = \frac{dh}{dz}$$

and using (1) we have:

$$\frac{d^2 y}{dz^2} - \frac{dy}{dz} \cdot \frac{d}{dz} \ln \gamma^2 + \gamma^2 y = 0 \quad (23)$$

Thus  $y$  obeys (1) if the W.K.B. criterion (6) is satisfied, but not otherwise. We must, in general, define the function:

$$y = \gamma W \quad (24)$$

and

$$\frac{d^2 W}{dz^2} + \beta^2 W = 0 \quad (25)$$

where

$$\beta^2 = \gamma^2 - \left( \frac{d}{dz} \ln \gamma \right)^2 + \frac{d^2}{dz^2} \ln \gamma \quad (26)$$

It appears then that  $W$  satisfies the Helmholtz equation (25) with a turning point at

$$\beta = 0 \quad (27)$$

i.e., at a different value of  $z$  from that corresponding to (s).

One might conclude from this that the function  $y = dh/dz$ , or some linear combination of  $h$  and  $y$ , changes character (from oscillating to exponential) at a different value of  $z$  than  $h$  itself. That such a con-

clusion is, in fact, quite specious is seen from the exact solutions (9), (10) and (12), (13). Mathematically, this apparent ambiguity arises from the feature that, at  $\gamma = 0$ ,  $\beta^2$  is singular: a proper discussion of (25) near the singularity must, needless to say, yield the same conclusion as examination of the exact solutions.

It follows also from what has been said above that the ray picture interpretation of  $\gamma, \beta$  is misleading near turning points. Conclusions concerning direction of phase travel, energy flux, levels of reflection, etc., that are based upon one's usual intuitive picture of the wave-number vector are, in the final analysis, only as good as the W.K.B. approximation; furthermore, inspection of (4) and (26) shows that the domain of validity of this approximation is precisely the same for  $h, y, w, dw/dz$  or for any linear combination of these functions.

We shall show in what follows that there are situations in geoaoustics and, in particular, in acoustic-gravity wave theory, for which these features of the wave solutions acquire a practical significance.

### 3. Phase velocity measurements in stratified media

We now focus our attention on the problem of measuring the  $z$  component of phase velocity (normal to the stratification), a problem one encounters in the use of vertical arrays in the oceans or atmosphere. This situation occurs, for example, whenever one attempts to interpret vertical phase velocities in terms of the stratification (structure) of the medium.

A general representation of a traveling harmonic wavefield is:

$$\phi = m e^{i\phi} e^{-i\omega t} \quad (28)$$

where  $m, \phi$  are real functions. The phase velocity in the  $z$  direction is:

$$v_z = \frac{\omega}{\partial \phi / \partial z} \quad (29)$$

If (2) is valid, it follows from (3) that:

$$\partial\phi/\partial z \approx \pm\gamma \quad (30)$$

and

$$v_z \approx \pm \frac{\omega}{\gamma} \quad (31)$$

Thus, if the W.K.B. approximation applies our usual understanding of vertical phase velocity as the ratio of angular frequency to vertical wavenumber component is valid: this is the realm of ray optics..

But if the conditions of validity for the W.K.B. method, i.e., inequalities (4) or (6), are not fulfilled, the statement (31) is no longer correct. This is easily seen by turning one's attention once again to the exact solutions for the linear  $\gamma^2$  case. Here  $\phi$  is given by the phase of  $H_{1/3}^{(1,2)}(s)$ , i.e.,

$$\phi = \arctan \left[ \cotan \frac{\pi}{3} - \frac{1}{\sin \frac{\pi}{3}} \cdot \frac{J_{-1/3}(s)}{J_{1/3}(s)} \right] \quad (32)$$

giving

$$\frac{\partial\phi}{\partial z} = \pm \frac{2\gamma}{\pi s} \frac{1}{J_{1/3}^2(s) + \frac{1}{3}[J_{1/3}(s) - 2J_{-1/3}(s)]^2} \quad (33)$$

By going to the limit of vanishing  $s$ ,  $\gamma$  (using equation 8), we have the vertical phase velocity at the turning point:

$$\text{as } \gamma \rightarrow 0, \quad v_z = \frac{\omega}{\partial\phi/\partial z} \rightarrow \pm 4\pi \cdot 3^{-4/3} |P|^{-1/3} [\Gamma(2/3)]^{-2} \omega \quad (34)$$

Thus the vertical phase velocity tends to a finite limit at the turning point (whereas literal acceptance of the ray picture leads to an infinite value for  $v_z$ ).

Therefore, unless exact solutions for  $\phi$ ,  $\partial\phi/\partial z$  are used, one may be appreciably in error in deducing the vertical phase velocity and angle of travel of a harmonic wavefront. Conversely, for measurements made near a

turning point interpretation of observed phase velocities may lead to erroneous conclusions concerning the stratification unless the exact equation (33) is used in ( ).

Figure 1 shows a plot of  $\phi$  and of  $d\phi/ds = \frac{1}{c} \cdot d\phi/dz$ . These quantities are plotted against  $s$ , so that the W.K.B. approximation for  $\phi$  is, by (2), a straight line of slope one; it's intercept is, by (16),  $s_0 = 5\pi/12$ . The exact curve for  $d\phi/ds$  are seen to agree with the W.K.B. values quite closely for

$$s \gg 1 \quad (35)$$

At  $s = 1$ , the W.K.B. error for  $d\phi/ds$  and, thus, for the critical phase velocity calculation is less than 5%: condition (35) as pointed out above, an alternative statement of (6).

It is of some interest to estimate (34) for low frequency acoustic waves (we will discuss acoustic-gravity waves in the next section). If one neglects the effect of gravity and substitute for  $P$  in (34) <sup>(1)</sup>:

$$P = \omega^2_0 \quad (36)$$

where  $c$  is the speed of sound.

This gives the turning point value:

$$\frac{v_z}{c} \approx 2.3 f^{1/3} \quad (37)$$

where  $f$  is the frequency in hz. In underwater acoustics one usually deals with gradients less than  $dc/dz \sim 1 \text{ sec}^{-1}$ . Thus  $2.3 f^{1/3}$  is pretty much the smallest value for (37) in marine acoustics experiments. It gives  $v_z/c \approx 5$  for  $f = 10 \text{ hz}$ , and  $v_z/c \approx 10$  for  $f = 10^2 \text{ hz}$ . In practical terms, the difference between an infinite move-out velocity and one that takes place at 10 times the speed of sound is hardly significant, when

one remembers that the measurement must take place over a fraction of a wavelength; in underwater sound experiments, for frequencies of  $10^2$  hz and above, this effect is unlikely to be important.

However, in infrasonic measurements in the atmosphere and thermosphere, the phenomenon is more likely to be observable. Thus, gradients of the order of  $dc/dz = 10^{-2} \text{ sec}^{-1}$  are characteristic of the thermosphere. Signals having periods as long as 4 minutes have been observed in connection with Apollo firings (10), i.e., we may take  $f \approx 4 \times 10^{-3}$  and  $v_{\infty}/c \approx 2$  at the turning point: here measurements of the critical phase velocity may yield misleading results, unless they are interpreted with the help of exact formulas of the type of (29), (33). As we shall see in Section 4, this effect is even more marked for acoustic-gravity waves in the 5 - 30 minute period bandpass at thermospheric heights.

As for the behavior of  $dh/dz$  near the turning point, we once again have recourse to the exact form for the progressive wave solution with linear  $\gamma^2$ , i.e., we use equation 15. Thus  $dh/dz$  corresponds to wavefields of the type:

$$\psi = n e^{i\psi} e^{-i\omega t} \quad (38)$$

where the phase  $\psi$  is, by (20):

$$\psi = \arctan \left[ \cotan \frac{\pi}{3} + \frac{1}{\sin \pi/3} \cdot \frac{J_{2/3}(s)}{J_{-2/3}(s)} \right] \quad (39)$$

For large  $s$ , i.e., in the region for which W.K.B. applies, it is clear from the usual asymptotic form for  $H_{2/3}^{(1,2)}(s)$ , that  $\psi \rightarrow s + \pi/12$  so that  $\partial\psi/\partial z \rightarrow \partial\phi/\partial z$  and the phase velocities of the  $\phi$ ,  $\psi$  fields are the same. But, for  $s < 1$ , equation (39) shows that the behavior of  $\partial\psi/\partial z$ ,  $\partial\phi/\partial z$  will be quite different. In fact, if we use the first terms of the power series for  $J_{\pm 2/3}(s)$ , equation (39) gives

$$\frac{d\psi}{ds} \approx 0.15 s^{1/3} \quad (40)$$

thus the phase velocity near the turning point goes to infinity like  $\gamma^{-2}$ . The behavior of  $\psi$ ,  $d\psi/ds$  is shown in Figure 1.

#### 4. Acoustic-gravity and internal gravity waves

We have established that one's interpretation of infrasonic wave behavior near turning points may be seriously in error, unless he uses the exact field equations. The use of ray-optical concepts here is erroneous; furthermore, the amount and sign of the error depends upon the field quantity being studied.

The largest errors will be encountered for acoustic-gravity waves at ionospheric heights. Measurements of traveling wave systems at those heights are currently a subject of considerable interest; naturally occurring wave systems, in particular, are being studied intensively for the light they shed upon upper atmospheric structure<sup>(11, 12)</sup>; dispersive acoustic-gravity waves with periods of several minutes have been detected by vertical phase sounders in connection with Apollo launches<sup>(10)</sup>

and disturbances in the same passband<sup>(13)</sup> as well as with longer periods<sup>(14, 15)</sup> have been seen in connection with large atmospheric explosions. It is therefore important to see how the considerations of Section 3 apply to acoustic-gravity and internal gravity waves at these heights.

With this in mind, let us fix our attention upon a specific model of a specific section of the thermosphere. Since much of the recent phase sounder work has been conducted with sounding frequencies in the 2 - 6 MHz passband, i.e., at true daytime heights in the 150 - 250 km range, let us select a reference height of 180 km and use this as the origin of our  $z$  coordinate. We may then expand  $\gamma^2$  into a Taylor series in  $z$ : keeping the first term only gives the linear law (7). The range

or validity of this model depends upon the actual rates of variation of the physical parameters. We begin with the differential equation for the elastic pressure  $p^{(14)}$ . Writing:

$$p = p_0^{1/2} h e^{i(ax - \omega t)} \quad (41)$$

gives equation 1, with

$$\gamma^2 = \frac{1}{c^2}(\omega^2 - \omega_{op}^2) + \frac{1}{v_x^2}(N^2 - \omega^2) \quad (42)$$

where  $c$  is the adiabatic sound speed,  $\rho$  the density,  $\omega_{op}$  is the elastic pressure "acoustic cutoff",  $N$  is the Brunt-Vaisälä frequency,  $v_x$  the horizontal phase velocity

$$v_x = \frac{\omega}{a} \quad (43)$$

It can be shown<sup>(14)</sup> that a reasonable fit for the actual  $\omega_{op}$ ,  $c$ ,  $N$  variations in the thermosphere is obtained by taking the perfect gas law with

$$c^2 = Pz + c_0^2 \quad (44)$$

i.e.,

$$N^2 = \frac{2}{c^2}(Pz + 0.4g) \quad (45)$$

$$\omega_{op}^2 = \frac{1}{4c^2}(g^2\tau^2 - P^2) \quad (46)$$

where  $\tau = 1.4$  is the ratio of specific heats. We may therefore approximate  $\gamma^2$  by a linear law as long as  $1/c^2$  is approximately linear, i.e., if

$$\frac{1}{c^2} \approx \frac{1}{c_0^2} \left(1 - \frac{Pz}{c_0^2}\right) \quad (47)$$

The exact solutions of Sections 2 and 3 will thus be useful if

$$\frac{Pz}{c_0^2} \ll 1 \quad (48)$$

Taking at  $z = 0$  (180 km height) the values<sup>(14)</sup>:

$$\begin{aligned} c_0^2 &= 5.94 \times 10^5 \text{ m}^2 \text{ sec}^{-2} \\ P &= 6.2 \text{ m sec}^{-2} \end{aligned} \quad (49)$$



We see that (48) imposes the limits:

$$|z| \ll 10^5 \text{ m} \quad (50)$$

In other words, we shall be limited to a range of about  $\pm 25$  km to each side of  $z = 0$ , i.e., to a zone of true heights between 155 km and 205 km. With this limitation in mind, we can apply the results of Sections 2 and 3 to the actual thermosphere.

Equations (45), (47), (49) give then:

$$N^2 = N_0^2 + nz, \quad n = -N_0^2 \frac{P}{c_0^2 z} \quad (51)$$

$$N_0 = 1.22 \times 10^{-2} \text{ rad sec}^{-1}$$

$$\frac{\omega_{op}^2}{c^2} = \frac{1}{c_0^2} \omega_{op_0}^2 + \Omega z, \quad \Omega = -2 \frac{P}{c_0^2 z} \cdot \frac{\omega_{op_0}^2}{c_0^2} \quad (52)$$

$$\omega_{op_0} = 0.682 \times 10^{-2} \text{ rad sec}^{-1}$$

Giving, in (7):

$$P = \frac{P}{c_0^2} \left[ 2 \frac{\omega_{op_0}^2}{c_0^2} - \frac{N_0^2}{v_x^2} - \frac{\omega^2}{c_0^2} \right] \quad (53)$$

$$\gamma_0^2 = \frac{\omega^2}{v_x^2} \left( \frac{v_x^2}{c_0^2} - 1 \right) + \frac{N_0^2}{v_x^2} - \frac{\omega_{op}^2}{c_0^2} \quad (54)$$

Using the numerical values (49) and (51), (52) we may calculate exact wave functions, phases, phase velocities etc., for  $155 \text{ km} < z < 205 \text{ km}$ . The procedure may be applied in the vicinity of a succession of  $z$  values to cover any desired height range in the thermosphere.

Figures 2 and 3 show the behavior of  $s$  at these heights, for the thermospheric constants (49), (51), (52): it is seen that the validity of the W.K.B. and ray-optical points of view is contingent upon frequency and  $v_x$ .

The vertical displacement  $\zeta$  is, to within a  $\pm i\omega$  factor, similar to the vertical velocity and is of particular interest since it is more directly related to the physical quantity measured in ionospheric phase sounders

(electron velocity). It is related to the elastic pressure field<sup>(14)</sup> by the equation:

$$\zeta = \left( -\frac{g}{v_x^2} p + \frac{\partial p}{\partial z} \right) / \rho \left( \omega^2 - \frac{g^2}{v_x^2} \right) \quad (55)$$

But

$$\frac{\partial p}{\partial z} = p \frac{1}{2} \frac{d}{dz} \ln \rho + \rho^{1/2} \frac{dh}{dz} e^{i(ax - \omega t)} \quad (56)$$

and

$$\zeta = \rho^{-1/2} \left( \omega^2 - \frac{g^2}{v_x^2} \right)^{-1} \left[ h \left( \frac{1}{2} \frac{d}{dz} \ln \rho + \frac{g}{v_x^2} \right) + \frac{dh}{dz} \right] \quad (57)$$

The relative importance of the  $h$ ,  $h_z$  terms near the turning point is determined by examining the exact solutions (14), (15) in the limit of  $\gamma \rightarrow 0$ ,  $s \rightarrow 0$ . It is then seen that  $dh/dz \approx 0.5 P^{1/3} h$ . But, for the case under consideration, for periods in the 10 - 30 minute range and for horizontal phase velocities  $v_x < 300 \text{ m sec}^{-1}$ ,  $P$  will be seen to vary in the  $10^{-13}$  to  $10^{-14} \text{ m}^{-3}$  range. Since furthermore  $g/v_x^2 > 10^{-4} \text{ m}^{-1}$  and  $d/dz \ln \rho \approx 10^{-5} \text{ m}^{-1}$ , we have, approximately:

$$\zeta \approx \rho^{-1/2} \left( \omega^2 - \frac{g^2}{v_x^2} \right)^{-1} \frac{g}{v_x^2} h \quad (58)$$

This approximation, good for  $v_x < 3 \times 10^2 \text{ m sec}^{-1}$  and periods in the 10 - 30 minute range, shows that we may usefully apply the exact solutions (14) to obtain the vertical phase velocity from (29) and (33) and, in particular, the limiting turning point value (34). Since we have here  $v_x^2/c^2 \ll 1$ , it follows from (53) that we have, approximately:

$$P \approx - \frac{N^2}{v_x^2} \frac{P}{c_0^2} \quad (59)$$

and the turning point phase velocity (34) is given by

$$\begin{aligned} \frac{v}{v_x} &\approx 4\pi \cdot 3^{-4/3} \left[ \Gamma\left(\frac{2}{3}\right) \right]^{-2} N^{-2/3} P^{-1/3} c_0^{2/3} \omega v_x^{-1/3} \\ &\approx 1.4 \times 10^3 \times \omega \times v_x^{-1/3} \end{aligned} \quad (60)$$

Thus, for periods of the order of 30 minutes and horizontal phase velocities  $\approx 10^2 \text{ m sec}^{-1}$  we have  $v_z/v_x \approx 1$  at the turning point, giving an apparent phase front inclination of  $45^\circ$ . Since  $v_z \approx v_x$  is often observed in the ionosphere<sup>(11)</sup>, these calculations are very pertinent to the interpretation of actual ionospheric observations of acoustic and acoustic-gravity travelling wavetrains.

It must be emphasized, however, that in general it is (57) that must be used: the interpretation of phase velocity measurements must then proceed by defining, with the help of (14) and (15), an exact phase function for  $h$ . This is slightly more involved than the calculations for our simplified case with  $v_x < 3 \times 10^2 \text{ m sec}^{-1}$ , but does not offer any particular difficulties.

## 5. Discussion

In problems involving wave propagation in stratified media, it is not difficult to convert a set of values for the local parameters and their derivatives into an explicit, simple criterion for the validity of the W.K.B. approximation in the solution of the fundamental Helmholtz equation (1). It is rather illuminating, in fact, to assume that  $\gamma^2$  in equation (1) may be treated locally as a linear function of  $z$  (a classic assumption first used by Langer<sup>(8)</sup> in his solution of the connection problem, the limitations of which are easily stated in terms of a permissible range of  $z$  values). This approach has the advantage of yielding locally exact solutions of (1) that are valid for simple turning points  $\gamma = 0$ , solutions which furthermore yield the W.K.B. approximations explicitly as an asymptotic limit. This shows that the W.K.B. method is valid in the limit of large enough phase paths, a statement that can be written very succinctly as  $s \gg 1$ , where  $s$  is given by (3). The W.K.B. approximation is essentially

the plane wave ray-optical point of view, so that this criterion yields also the domain of validity for the usual formulas relating phase velocity (vertical phase velocity, in the present case) to the corresponding wave-number components and angles of incidence. In fact, a glance at Figure 1 shows that the criterion may be, in practice, relaxed to  $s > 1$ ; in this domain everything can be interpreted in terms of the usual ray-optical imagery. In the vicinity of a turning point of the Helmholtz equation, interpretation becomes more delicate: the W.K.B. approximation is no longer valid and the usual concepts of wavenumber and phase velocity components fail us - but we do, in this case, have exact equations to fall back on. The failure of the simple-minded ray-optical concepts here is best understood as the result of interference effects concentrated near the turning point. Thus, we may visualize the flow of wave energy through a continuously stratified medium as actually taking place through a succession of very thin homogeneous layers, representing incremental changes in the physical parameters. Assume, for the purpose of simple visualization, that we are dealing with acoustic waves in a temperature stratified medium; when the wavefronts and wavenumber vectors can be unambiguously defined they are mutually orthogonal, and the energy flux vector is colinear with the wavenumber: as the wave travels towards regions of higher sound velocity, the rays (energy paths) are bent according to Snell's law; as the wave progresses reflections take place at the boundary of each incremental layer: as the layers become infinitesimal, this provides a continuous reflection process, a sort of effective backscatter of the energy. As has been pointed out<sup>(7)</sup>, the W.K.B. approximation neglects this backscatter entirely, a procedure that is justified as long as the parameters of the medium vary little over a vertical wavelength. But then, as we approach a turning point, we near a condition of grazing incidence so that, no matter how small the increment

in sound velocity between neighboring layers, one eventually approximates a condition of total reflection: the backscatter is then no longer small and cannot be neglected. This is the physical reason for which the W.K.B. method fails; it is also the reason for which it becomes impossible to define a vertical phase velocity: even an isolated "upward" or "downward" moving wavefield of the type (19) or (20) will thus get involved with interference effects near  $\gamma = 0$ . These effects are irreducible and cannot be eliminated by directional filtering. It must be emphasized that this is not the same thing as the generation of a standing wave pattern such as (9) (e.g., with  $A = B$ ) by a superposition of the incident and reflected wavefields (19), (20), a pattern that is, in principle, analyzable into two separate upward and downward moving wavefields. It is, for instance, possible to produce a pure wavefield of the type  $H_{1/3}^{(1)}(s) e^{-i\omega t}$  moving away from the turning point (e.g., by leakage of a harmonic wave system through a high velocity barrier); such a wave would be carrying energy away from the turning point but would, nevertheless, scatter and interfere with itself in a narrow region near  $\gamma = 0$ . Since this scatter actually consists of multiple reflections<sup>(2,7)</sup>, it contains both upward and downward moving components, so that directional filtering cannot get rid of these effects which are thus irreducible.

This, then, is the root of the difficulties one may have in interpreting the phase velocities of a travelling harmonic wavefield in the vicinity of turning points in stratified media, i.e., in regions for which the W.K.B. method fails. We have, in Sections 3 and 4, dealt with these questions explicitly in the case of infrasonic and acoustic-gravity wave propagation in the thermosphere. In the last few years, observations have been made of travelling wave systems at these heights and, in some cases<sup>(11)</sup>, the observations appear to fall in the range of values for which interpretation

becomes ambiguous. The purpose of this paper has been, chiefly, to explore and clarify the nature of the ambiguity in such a way that future experimental measurements at infrasonic and acoustic-gravity wave frequencies will be able to deal with it; since this is largely a matter of making sure that, in such instance, one has enough vertical measurement points to determine whether or not he is approaching a turning point region, this should not be difficult to achieve.

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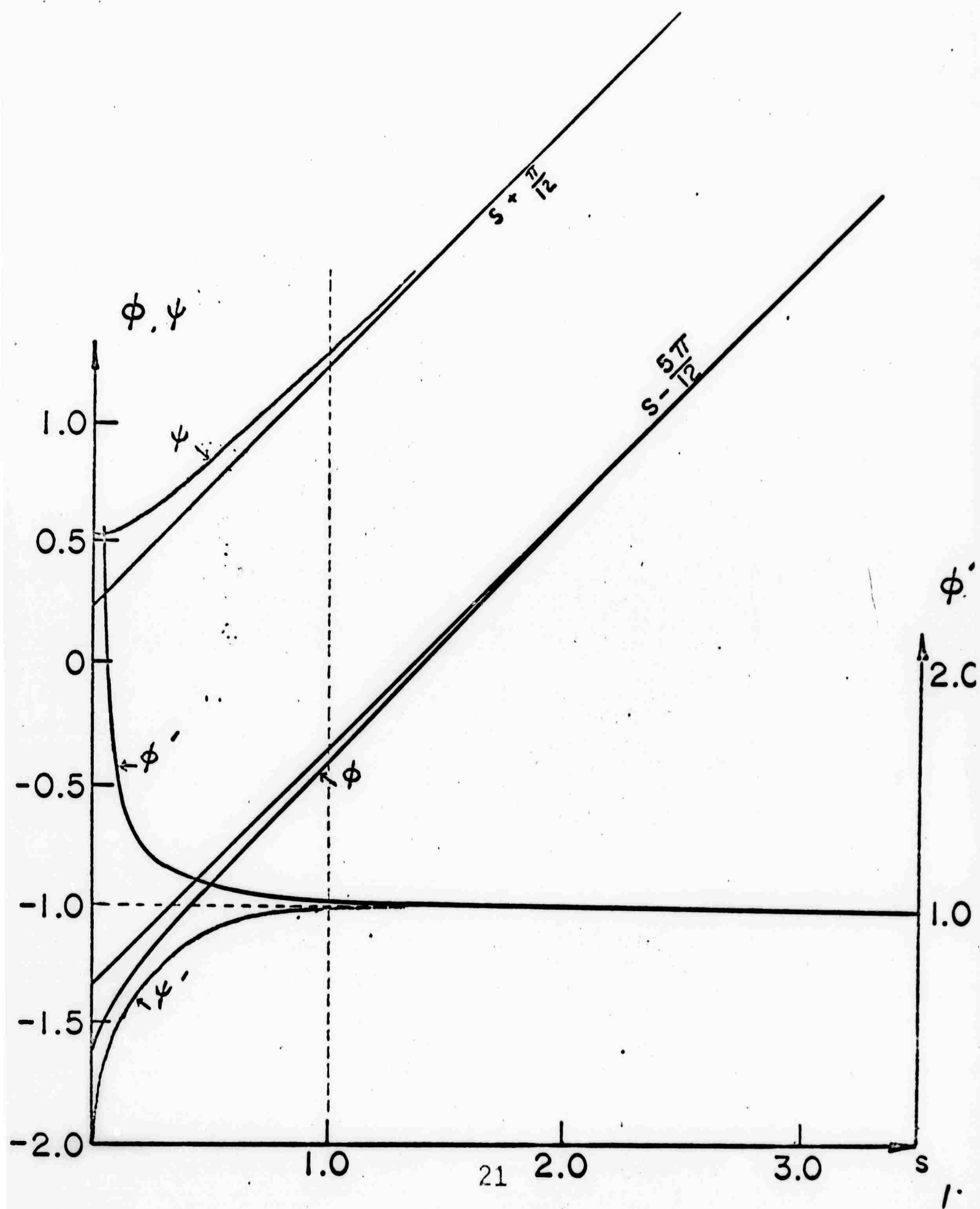
### Figure Captions

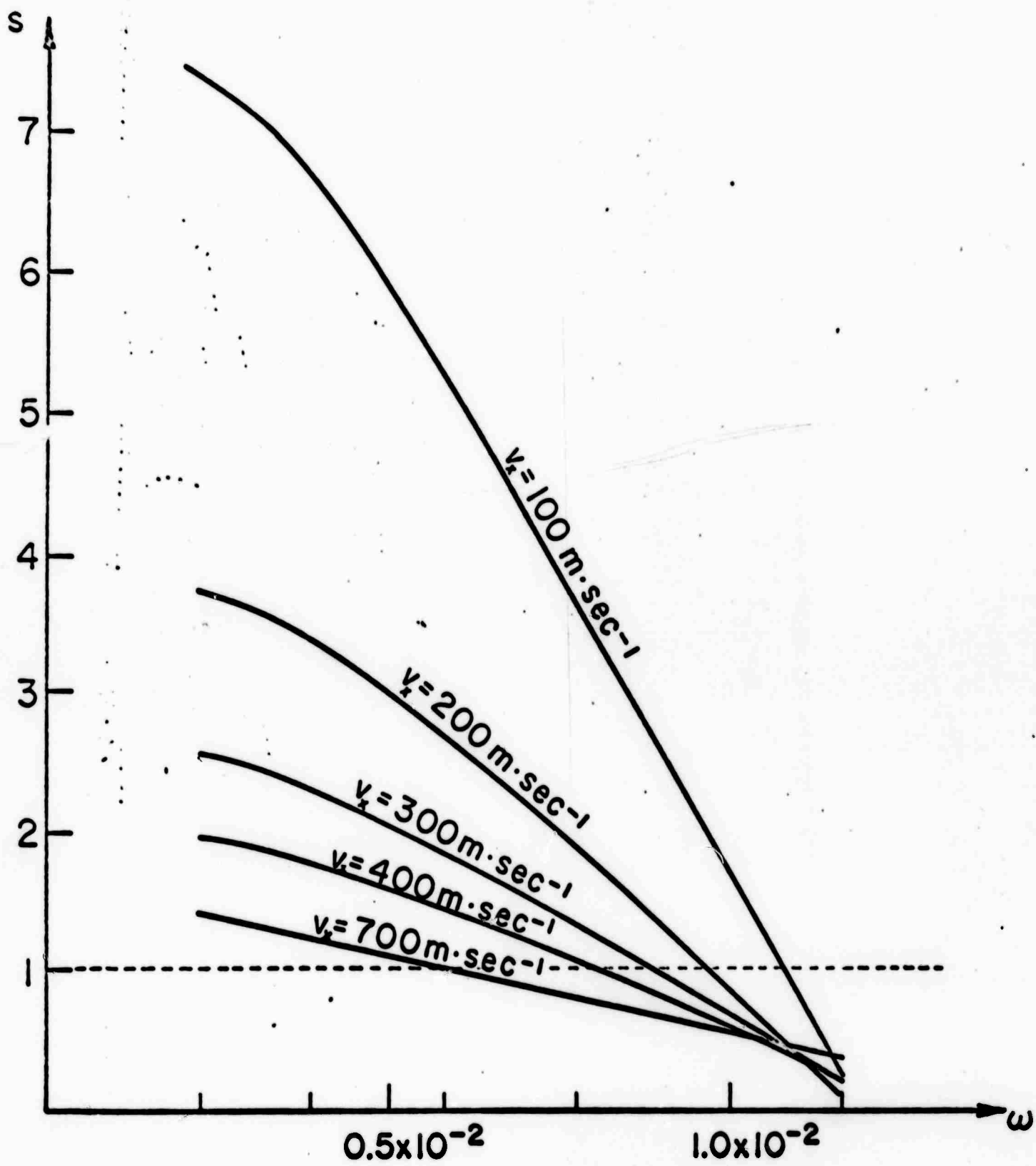
Figure 1. Progressive wave phase angles as a function  $s$ ;  $\phi$  is the phase of  $h$ ,  $\psi$  the phase of  $dh/dz$  in a medium of linear  $\gamma^2(z)$ . The asymptotes of  $\phi$ ,  $\psi$  correspond to the W.K.B. approximation. It is seen that the slopes  $\phi'$ ,  $\psi'$  have quite different behavior near  $s = 0$  (turning point), implying different limiting vertical phase velocities. It is also seen that phase and phase velocity are satisfactorily given by W.K.B. for  $s \geq 1$ .

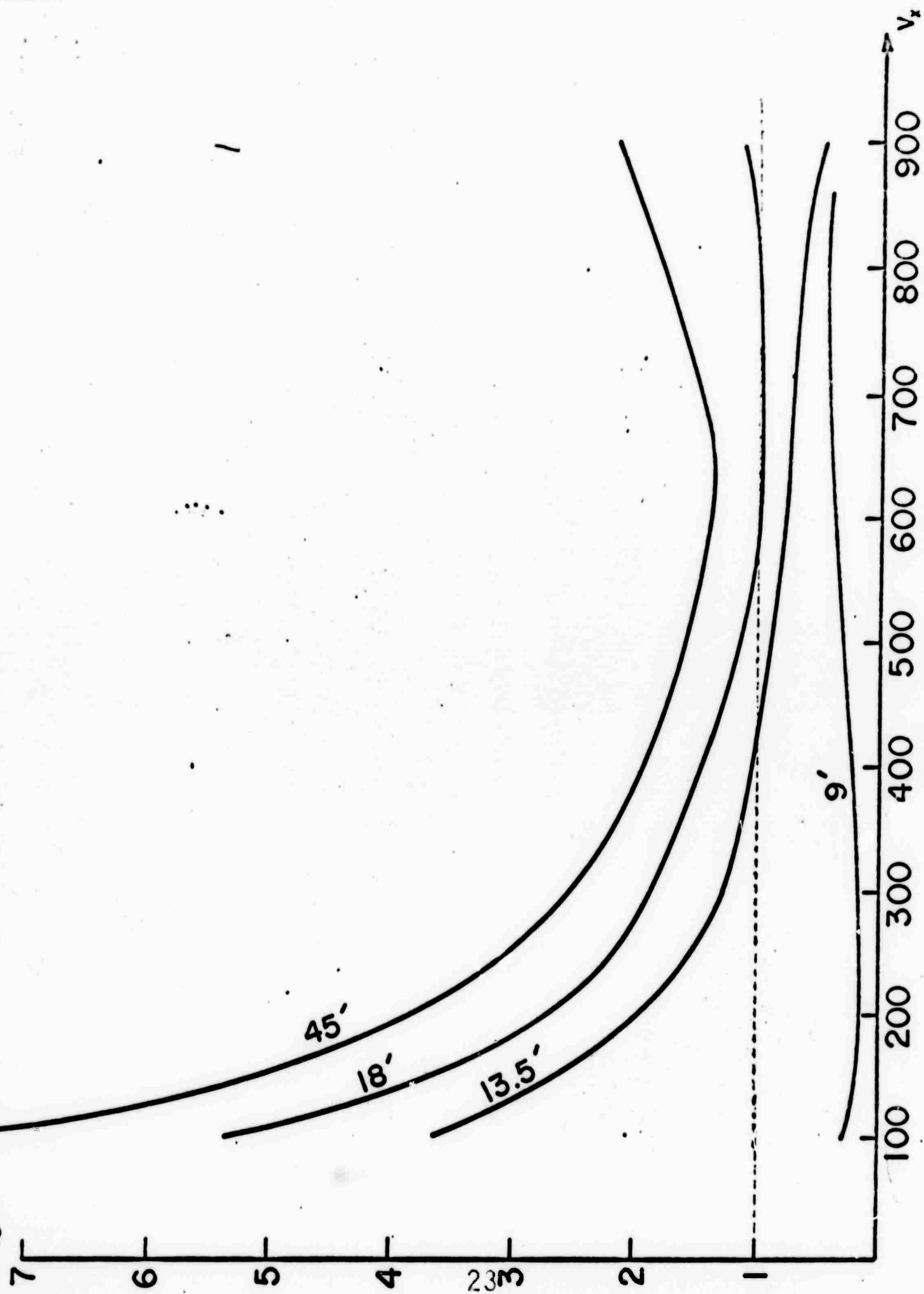
Figure 2. Behavior of  $s$  in the earth's thermosphere at an altitude of 180 km, as a function of  $\omega$  for various selected values of the horizontal phase velocity  $v_x$ . The dotted line  $s = 1$  separates, for all practical purposes, the region in which the W.K.B. approximation is valid ( $s > 1$ ) from the region in which it is insufficiently accurate for the calculation of phase velocity ( $s < 1$ ); note that, in the latter region, it may still be possible to use W.K.B. for order of magnitude amplitude calculations.

Figure 3. Same as Figure 2, but with horizontal phase velocity  $v_x$  as variable and period in minutes as parameter.









DAMPING OF VERY LONG INTERNAL GRAVITY  
WAVES IN THE ATMOSPHERE

Classic waveform calculations for atmospheric explosions, such as those of Harkrider (1964), have been largely (but not entirely, see e.g., Harkrider and Wells, 1968) limited to the acoustic and shorter period internal gravity wave spectrum (periods  $T < 5$  min.). These calculations reproduce quite well the waveforms observed on microbarographs. There is now a growing body of evidence indicating that long wavelength disturbances generated by nuclear explosions in the atmosphere can also travel to great distances. Both ionospheric measurements (Breitling et al, 1967; Rose et al, 1961; Hultquist et al, 1961; Dieminger et al, 1962; Herron and Montes, 1970) and long period barographs (Tolstoy and Herron, 1970; Herron and Montes, 1970) indicate the occasional existence of long wavelength arrivals having travelled at least once around the globe. Propagation to these great distances implies that, for long wavelengths, attenuation need not be prohibitive and that channelling and waveguide effects must take place. Although it is possible that the surface gravity mode ( $m = 0$ ) may account for some of these observations (Tolstoy and Herron, 1970; Tolstoy and Pan, 1970), multiply reflected internal gravity waves are potentially capable of explaining arrivals with group velocities below  $600 \text{ m.sec}^{-1}$ . Thus we shall show in what follows that the attenuation of the lower modes ( $m = 1, 2, 3$ ) need not be excessive. The relatively short range observations of Stoffregen et al (1961) can be explained by internal gravity waves (Hines, 1967) along direct ray paths only, traveling from the explosion source to the observation point, with no reflections. It has been suggested that these waves are absorbed in the upper atmosphere because of what amounts to an indefinite increase in kinematic viscosity with height as the density tends to zero. In other words, upwards traveling energy is attenuated and transformed into heat: there is no reflected wave

energy. This situation would correspond schematically to that of figure 1. As we shall see, this mechanism is only plausible for sufficiently short wavelengths: long waves are reflected in the manner demonstrated below, an effect which was first shown by Yanowitch (1967). This appears to be the only reasonable explanation for the observed propagation of long wavelengths to ranges of  $10^4$  km or more.

In order to keep the discussion simple, we shall consider a locally isothermal atmosphere, i.e., an atmosphere consisting of layers in which:

$$\rho = \rho_0 e^{-2\nu z} \quad (1)$$

where  $\nu$  is a constant. At ionospheric heights between 200 and 500 km.,  $\nu$  is probably such that:

$$1 \times 10^{-5} \text{ m}^{-1} \leq \nu \leq 3 \times 10^{-5} \text{ m}^{-1} \quad (2)$$

Assuming then

$$\nu = 1.5 \times 10^{-5} \text{ m}^{-1} \quad (3)$$

implies that the thickness  $d$  over which the density varies by one order of magnitude (the scale height) is:

$$d \approx 2.3/2\nu = .8 \times 10^5 \text{ m} \quad (4)$$

Essentially complete absorption of the energy should take place for wavelengths of the order a few  $d$  and less: this covers, in practice, a good part of the observed TID spectrum.

The statement that the energy of an upward traveling wave gets entirely absorbed and changed into heat hinges on a literal acceptance of the equations of continuum mechanics and upon the ensuing infinite

kinematic viscosity as  $\rho \rightarrow 0$ . This statement must also be qualified in terms of the wavelength: it has been shown by Yanowitch (1967) that waves of sufficient length are reflected; Yanowitch interprets this result to mean that one has reflections from layers of large, rapidly varying kinematic viscosity. Yanowitch's proof is based upon a straightforward discussion of the effects of the viscous terms on the internal gravity wave field equations.

In fact, of course, the continuum equations cease to be valid as  $\rho \rightarrow 0$ , and a more sophisticated treatment is required: needed here is an adequate molecular model of the gas, combined with an analysis of Boltzmann's equations for a rarefied, density stratified gas in a gravity field. Such an analysis would be extremely difficult and is beyond our reach at present. However, pending the appearance of a more rigorous theory along such lines, one may perhaps indulge in some assumptions as to what the long wavelength limit of such solutions is likely to show (Tolstoy, 1967). Interestingly enough, it is possible in this manner to obtain a result comparable to Yanowitch's.

We assume the critical dimensionless parameter in this problem to be  $k\ell$ , where  $k$  is the wavenumber and  $\ell$  is the mean free path of the molecules. For angles that are not too steep, it would seem plausible to assume that:

- 1) For heights such that

$$k\ell \ll 1 \quad (5)$$

the equations of continuum mechanics will apply

- 2) For heights such that

$$k\ell > 1 \quad (6)$$

i.e., if the mean free path of the molecules is of the order of or less

than the wavelength, the medium acts effectively as a vacuum [providing hydromagnetic interactions can be neglected in the atmosphere at these heights: this is probably so for periods less than three hours or so, as can be shown by application of the Dungey-Fejer-Hines criterion, see e.g., Tolstoy, 1967]. An equivalent statement would be that, for collision times longer than the characteristic time scale (period) of the disturbance, no transmission of energy can occur (this is, effectively, the operational definition of a vacuum).

3) Between the vacuum region defined by (6) and the continuum corresponding to (5) there exists a zone of high kinematic viscosity in which most of the attenuation takes place. We shall refer to this as the transition layer. The height of this layer above the earth's surface depends upon the wavelengths considered: according to our best estimate of the mean free path, the base of the layer should be at a height of the order of  $500 \pm 50$  km. for wavelengths in the  $5 \times 10^2 - 5 \times 10^3$  km. band.

Kinetic gas theory tells us that the kinematic viscosity  $\eta$  is given approximately by:

$$\eta \approx \frac{1}{3} \langle v \rangle \ell \quad (7)$$

where  $\langle v \rangle$  is the mean velocity of the molecules and  $\ell$  is their mean free path. The same theory tells us that

$$\langle v \rangle \approx c \quad (8)$$

where  $c$  is the adiabatic sound speed.

Elementary wave theory further shows that, for constant  $\eta$ , a progressive harmonic wave in a continuous medium attenuates like  $e^{-qx}$ , where

$$q \approx \eta k^2 / v_{ph} \quad (9)$$

$v_{ph}$  being the phase velocity. If, for an order of magnitude argument, we assume that  $v_{ph}$  and  $c$  are of the same order, i.e.

$$v_{ph} \approx c \approx \langle v \rangle \quad (10)$$

and

$$q \approx \frac{1}{3} (k\ell)k \quad (11)$$

thus, for fixed  $k\ell$ , as  $k \rightarrow 0$ ,  $q \rightarrow 0$ .

Let us then assume we have free space for

$$k\ell > 1 \quad (12)$$

and a continuum for

$$k\ell \leq 10^{-1} \quad (13)$$

This is tantamount to the statement that, for given  $k$ , the attenuation takes place primarily within a layer in which  $\ell$  varies by one order of magnitude; since  $\rho \propto \ell^{-1}$ , it follows that the thickness of the transition layer is essentially the scale height  $d$  (equation 4).

Finally, we make the additional assumption that, within this transition layer, the attenuation for very long wavelengths is similar to that of a layer of thickness  $d$  having a mean kinematic viscosity  $\bar{\eta}$ :

$$\bar{\eta} = \eta_0 \frac{1}{d} \int_0^d e^{2\nu z} dz \approx 4\eta_0 \quad (14)$$

where  $\eta_0$  is the kinematic viscosity at the bottom of the transition layer.

In this picture, a long wavelength ( $k \rightarrow 0$ ) internal gravity wave traveling upwards suffers some attenuation as it passes through the transition layer, but is then partially reflected (figure 2): it can be plausibly assumed that, for long enough wavelengths, this reflection process is similar to that obtaining at a "free" surface of contact between a continuous fluid and a vacuum. In the present model, interposed between the vacuum and the continuous fluid is the transition layer; the thickness of this layer is of the order of a scale height and is, in an isothermal model of the upper atmosphere, independent of the wavelength: for very



long wavelengths, therefore, it acts effectively as a thin high viscosity film.

The validity of these assumptions cannot be demonstrated in any rigorous way on theoretical grounds. What we have done is simply to make some heuristic assumptions concerning the possible behavior of solutions of the Boltzmann equation for wavelengths long compared to d.

If these qualitative arguments are accepted, it follows that one can estimate the loss of amplitude for a long wavelength wave upon reflection from the transition layer. We are limited to the case

$$\gamma d \leq 1 \quad (15)$$

i.e., where  $\gamma$  is the z component of the wavenumber: we must have wavelengths that are at least  $2\pi$  times the thickness of the transition layer. Furthermore, in order to compute roughly the decay of the wave as it travels through this layer in both directions, we assume an attenuation like  $e^{-qL}$ , where  $L$  is the effective path. For order of magnitude arguments we may assume, in this part of the frequency-wavenumber domain, moderate angles of incidence and x,z wavenumbers of the same order. We thus take  $k \approx \gamma$  with a total effective path length of the order of  $4d$ , i.e.,

$$qL \approx 4dq \approx \frac{4}{3} \gamma^2 \gamma d \quad (16)$$

Thus the reflection coefficient modulus would be:

$$|R| \approx e^{-qL} \approx e^{-4/3 \gamma^2 \gamma d} \quad (17)$$

Remarkably enough, if we take  $\gamma d \approx \frac{3}{4}\pi$  (which is consistent with equation 12), we get:

$$|R| \approx e^{-\pi \gamma d} \quad (18)$$

This is the result obtained by Yanowitch (1967) from the continuum equations, for the reflection of internal gravity waves from a region of large kinematic viscosity. Considering the relationship between kinetic

theory and continuum mechanics, this is not entirely surprising. We have simply confirmed the statements that most of the attenuation comes from a layer in which the kinematic viscosity becomes prohibitive, that the thickness of this layer is of the order of a scale height  $d$  and that for wavelengths much greater than  $d$  reflection takes place. For very long wavelengths, (17) or (18) show that this reflection becomes total.

For modest amounts of attenuation, then, we write:

$$|R| = 1 - \epsilon \quad (19)$$

with

$$\epsilon = qL \approx \pi \gamma d \quad (20)$$

If now we consider a waveguide of thickness  $h$  in which multiple imperfect reflections of this type will occur, then the attenuation in the horizontal direction goes like  $e^{-\delta r}$ , where (Tolstoy and Clay, 1966):

$$\delta \approx \epsilon / 2h \tan \theta \quad (21)$$

where it is assumed that no losses other than those due to reflection at the top surface take place.

It is perhaps more convenient to obtain an expression for the attenuation in terms of the wavenumber and frequency. For this we return to (9). Using, as in (16),  $L = 4d$  we have:

$$qL \approx \epsilon \approx 4\bar{n} \frac{k^3}{\omega} d \quad (22)$$

Thus

$$\delta \approx 2\bar{n} \frac{d}{h} \frac{k^3}{\omega} \frac{1}{\tan \theta} \quad (23)$$

assuming modest  $\theta$ , i.e.,  $\tan \theta \approx 1$ , we have:

$$\delta \approx 2\bar{n} \frac{d}{h} \frac{k^3}{\omega} \quad (24)$$

i.e., the attenuation is given by the usual factor, weighted by the ratio of the path length in the highly viscous layer to the total path length.  $\bar{n}$  is given by (14), and it is not easy to pick a suitable figure for  $n_0$ .

The graph given by Midgley and Liemohn (1966) gives  $\eta = 10^6 \text{ m}^2 \text{ sec}^{-1}$  at  $z = 200 \text{ km}$ . It seems, though, that the molecular viscosity decreases with height, somewhat counteracting the  $1/\rho$  effect. This suggests something of the order of  $10^6 \text{ m}^2 \text{ sec}^{-1}$  at heights in the 400-500 km range. On the other hand, if we take an elementary kinetic gas theory result for a small hard sphere model, we find (Tolstoy, 1967) for the mean free path in meters:

$$\ell = 6.2 \times 10^{-8} \rho^{-1} \quad (25)$$

and (7) gives

$$\eta = \frac{1}{3} \langle v \rangle \times 6.2 \times 10^{-8} \rho^{-1} \quad (26)$$

assuming  $\langle v \rangle = c = 8 \times 10^2 \text{ m. sec}^{-1}$  and  $\rho = 10^{-11} \text{ kg. m}^{-3}$  at the base of the transition layer, somewhere near the 500 km level,

$$\eta_0 = 1.8 \times 10^6 \quad (27)$$

Take then, in (24)

$$\begin{aligned} \eta_0 &= 1.4 \times 10^6 \\ \bar{\eta} &= 5.6 \times 10^6 \end{aligned} \quad (28)$$

$$\frac{d}{h} \sim .15$$

and we have

$$\delta = 1.7 \times 10 \frac{k^3}{\omega} \quad (29)$$

Assuming a phase velocity in the neighborhood of  $600 \text{ m. sec}^{-1}$ , wavelengths of the order of  $6 \times 10^2 \text{ km}$ . will propagate to ranges of the order  $10^4 \text{ km}$  with an amplitude decay of .1. Thus, with the numbers we have used, only wavelengths in excess of  $6 \times 10^2 \text{ km}$  will propagate to any distance.

These very simpleminded calculations suggest that the waveguide mode picture for internal gravity waves could be valid for long enough wave-

lengths. As is well known, it also applies for that part of the short wavelength spectrum for which the energy finds itself trapped below heights of 200-300 km (see e.g., Harkrider, 1964; Harkrider and Wells, 1968). It seems therefore that a good part of the relevant explosion-generated internal gravity waves can be so treated. However, it must always be kept in mind that, ultimately, everything hinges on the magnitude of  $\eta$ : satellite drag measurements at these heights suggest that this quantity will vary a great deal, depending upon solar activity and other factors (Harris and Priester, 1967; Schilling, 1967). One may thus expect the actual attenuation to vary substantially (and in both directions) from the above estimates, which can only be regarded as plausible mean values.

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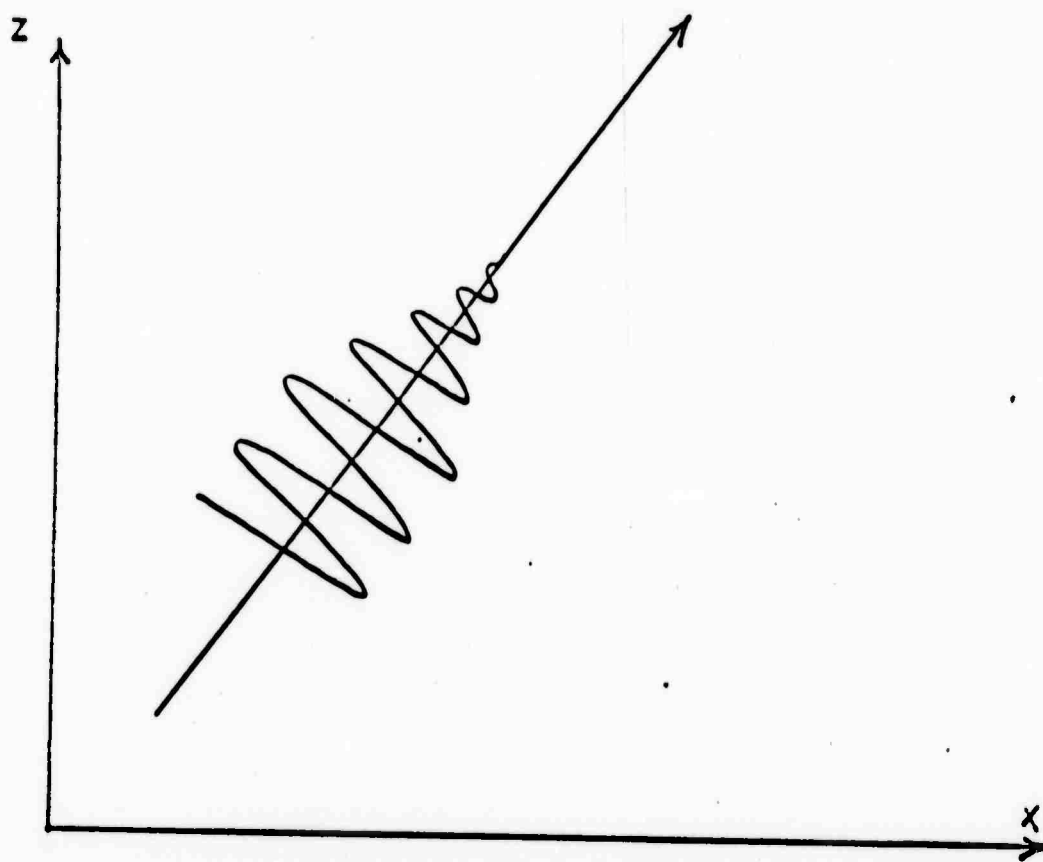
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# FIGURE CAPTIONS

Figure 1: Complete absorption of an upward traveling internal gravity wave in an isothermal layer with kinematic viscosity tending to infinity as  $z \rightarrow \infty$ .

Figure 2: Reflection with loss of amplitude by transition layer of high viscosity but thin compared to the wavelength.





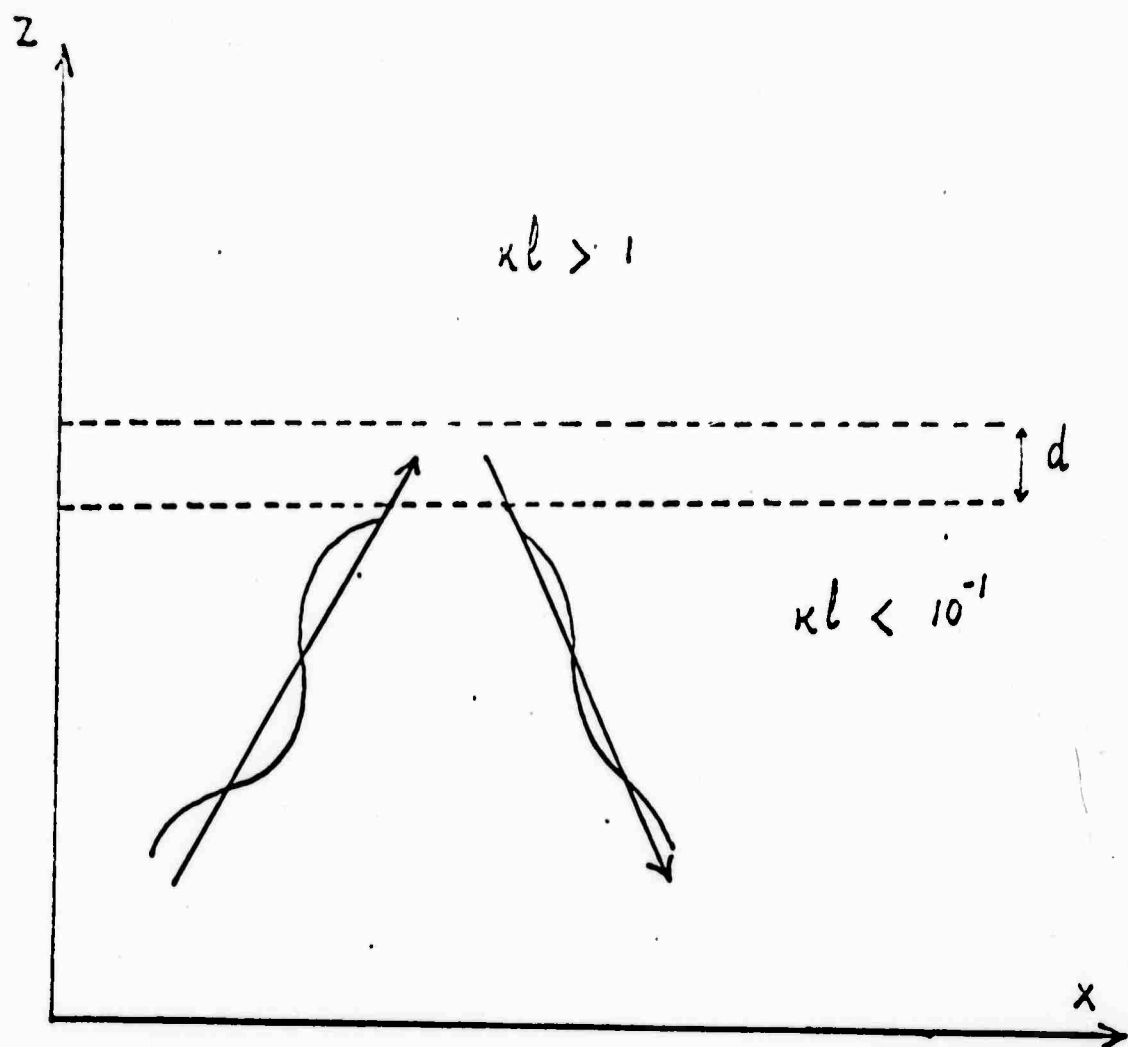


FIG. 2.6